Journal of Statistical Physics, Vol. 103, Nos. 1/2, 2001

On Stable Oscillations and Equilibriums Induced by Small Noise

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Received May 2, 2000; final October 18, 2000

We consider the motion of a light particle when the force field is perturbed by a small nose. If a certain relation between the mass of the particle and the noise intensity holds, the motion of the particle will be close to periodic oscillations or to a stable equilibrium which do not exist without the noise. We study various classes of random perturbations. In particular, we consider the question of computer simulation of these effects and calculate the correction term which appears when the Gaussian perturbations are replaced by the simple random walk. These are the stochastic-resonance-type effects, and their mathematical description is based on the large deviation theory.

KEY WORDS: Stochastic resonance; noise-induced oscillations and equilibriums; large deviations; perturbed systems.

1. INTRODUCTION

Consider a differential equation

$$\delta \ddot{Y}_t = f(\dot{Y}_t, Y_t), \quad Y_0 = y, \quad \dot{Y}_0 = x, \quad x, y \in \mathbb{R}^1, \quad 0 < \delta \ll 1 \tag{1}$$

Suppose that the set $\mathscr{E} = \{(\dot{y}, y) : f(\dot{y}, y) = 0\}$ consists of a curve $y = Y(\dot{y})$ as shown in Fig. 1, and the function $f(\dot{y}, y)$ is positive below this curve and negative above it. Then, as is known, for $0 < \delta \ll 1$, the solution will perform oscillations, which are close to the counterclockwise rotations around the loop $ABCD$ shown in Fig. 1: It takes time of order 1 to go form B to C and from D to A, and transitions between A and B and between C and D occur in time $o(1)$ as $\delta \downarrow 0$. These oscillations appear due to the bifurcations in the one-dimensional vector field $f(x, y)$, y is a parameter, when y changes.

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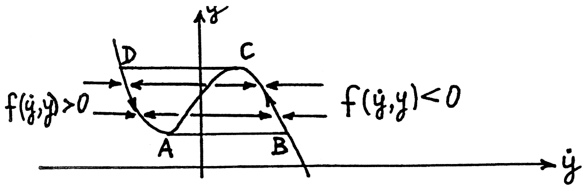

Fig. 1.

Let now the set $\mathscr{E}$ consist of three smooth curves $X^*_-(y)$, $X^*_0(y)$, $X^*_+(y)$ situated in the plane $(\dot{y}, y)$ as shown in Fig. 2. Let $X^*_-(y)$ and $X^*_+(y)$ be stable zeros of the field $f(x, y)$ for any $y \in \mathbb{R}^1$. Assume that

$$X^*_-(y) < X^*_0(y) < X^*_+(y)$$

for $y \in \mathbb{R}^1$. Then the system (1) has no periodic solutions, and if, additionally, we assume that the axis y does not intersect with these curves, the system has no equilibrium points.

Consider small random perturbations of Eq. (1). To be specific, I will consider first white-noise-type perturbations:

$$\begin{aligned} \delta \ddot{y}^\varepsilon_t &= f(\dot{y}^\varepsilon_t, y^\varepsilon_t) + \sqrt{\varepsilon}\, \sigma(\dot{y}^\varepsilon_t, y^\varepsilon_t) \circ \dot{W}_t \\ y^\varepsilon_0 &= y, \qquad \dot{y}^\varepsilon = x, \qquad 0 < \varepsilon \ll 1 \end{aligned} \tag{2}$$

Here W_t is the one-dimensional Wiener process, $\sigma(x, y)$ is a bounded separated from zero, smooth function. The stochastic term in (2) is understood in the Stratanovich sense.

The goal of this paper is to show that the perturbed system (2) and similar multidimensional systems under certain relation's between δ and ε may have periodic, in a sense, solutions and stable equilibriums even if $f(\dot{y}, y)$ behaves like in Fig. 2. If the system behaves like in Fig. 1, other

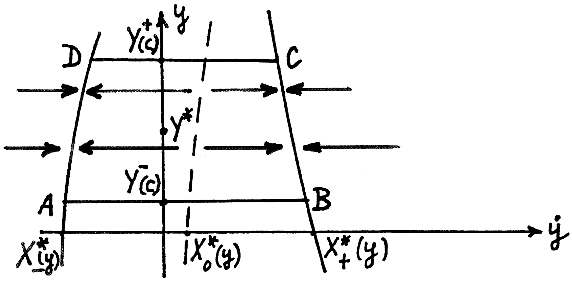

Fig. 2.

oscillations and a "stable equilibrium point" may appear. The velocity of the particle governed by Eq. (1) has, say in the case of Fig. 2, two stable equilibriums $X^*_+(y)$ and $X^*_-(y)$, $y \in \mathbb{R}^1$. If the system is perturbed by a small noise, then in the generic case, one of these equilibrium velocities (which depends on the position of the particle) is "more stable" than the other (see ref. 3, Chap. 4), so that the particle, roughly speaking, moves with this "more stable" velocity until the other equilibrium velocity becomes "more stable." This switching results in the oscillation or stabilization studied in the paper. One should underline that the period of those oscillations and the rate of convergence to the equilibriums are of order 1 as ε, $\delta \downarrow 0$. Note also, that our approach works in a rather general situation. In particular, we do not assume that the system under consideration is potential: one can introduce the notion of quasi-potential (see ref. 3, Chap. 4) for a wide class of dynamical systems and random perturbations.

This is a stochastic-resonance type effect (see refs. 1, 2), and the mathematical treatment of this effect is based on the large deviation theory for stochastically perturbed dynamical systems.[3] Therefore, there are, actually, much fewer reasons to consider Gaussian perturbations: the belief that the noise is Gaussian is based on the central-limit-theorems-type results. But the Gaussian approximation, in general, does not work for large deviations. So that we have to study other types of perturbations. We will see that, although the qualitative results are similar in these cases, the numerical characteristics are different.

In the next section we study Eq. (2): we present the results and give a sketch of the proof in the case of white-noise-type perturbations. Then, in Section 3, we consider some other classes of perturbations. In particular, shot-noise-type perturbations and rapidly oscillating perturbations are considered. Questions related to computer simulations of the effects which appear in system (2) when δ and ε are small, are examined in Section 3 as well. In the last section, systems with many degrees of freedom are considered.

2. ONE DEGREE OF FREEDOM

Rewrite (2) as the system:

$$\begin{aligned} \dot{X}^\varepsilon_t &= \frac{1}{\delta} f(X^\varepsilon_t, Y^\varepsilon_t) + \frac{\sqrt{\varepsilon}}{\delta} \sigma(X^\varepsilon_t, Y^\varepsilon_t) \circ \dot{W}_t \\ \dot{Y}^\varepsilon_t &= \dot{X}^\varepsilon_t, \qquad X^\varepsilon_0 = x, \qquad Y^\varepsilon_t = y \end{aligned} \tag{3}$$

Together with (3), consider the equation for the X-component with a frozen variable $Y = y$:

$$\dot{X}_t^{\varepsilon, y} = \frac{1}{\delta} f(X_t^{\varepsilon, y}, y) + \frac{\sqrt{\varepsilon}}{\delta} \sigma(X_t^{\varepsilon, y}, y) \circ \dot{W}_t$$
$$\dot{X}_0^{\varepsilon, x} = x \tag{4}$$

We say that Condition 1 is satisfied if for any $y \in \mathbb{R}$, the set $\{x \in \mathbb{R} : f(x, y) = 0\}$ consists of three points $X_-^*(y) < X_0^*(y) < X_+^*(y)$. The functions $X_\pm^*(y)$, $X_0^*(y)$, as well as $f(x, y)$ are assumed to have bounded and continuous first derivatives. The points $X_\pm^*(y)$ are stable roots of the equation $f(X_\pm^*(y), y) = 0$ for any $y \in \mathbb{R} : f'_x(X_\pm^*(y), y) < 0$, $y \in \mathbb{R}$. The function $\sigma(x, y)$ in (3) is assumed to have bounded derivatives and $\sigma^2(x, y) \geqslant \sigma_0^2 > 0$.

A typical example of the functions $X_\pm^*(Y)$, $X_0^*(y)$ satisfying Condition 1 is shown in Fig. 2.

If Condition 1 is satisfied, we can define functions

$$V_\pm(y) = -2 \int_{X_\pm^*(y)}^{X_0^*(y)} \frac{f(z, y)}{\sigma^2(z, y)} \, dz$$

for any $y \in \mathbb{R}$.

We say that Condition 2 is satisfied if the functions $V_\pm(y)$ defined above are monotone,

$$V'_+(y) < 0, \qquad V'_-(y) > 0, \qquad \bar{V}_\pm = \inf_{y \in \mathbb{R}} V_\pm(y) > 0$$

and for some y^* and Λ: $V_+(y^*) = V_-(y^*) = \Lambda$ (see Fig. 3).

For $c > (\bar{V}_+ \vee \bar{V}_-)$, define $Y_-(C)$ $(Y_+(C))$ as the solution of the equation $V_-(Y_-(C)) = C$, $(V_+(Y_+(C)) = C)$.

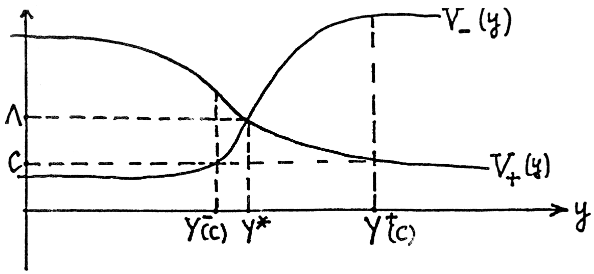

Fig. 3.

Let $\bar{Y}^y_+(t)$ and $\bar{Y}^y_-(t)$ be defined as the solutions of equations:

$$\begin{aligned} \dot{\bar{Y}}^y_+(t) &= X^*_+(\bar{Y}^y_+(t)), \qquad \bar{Y}^y_+(0) = y \\ \dot{\bar{Y}}^y_-(t) &= X^*_-(\bar{Y}^y_-(t)), \qquad \bar{Y}^y_-(0) = y \end{aligned} \tag{5}$$

Put

$$T_1(C) = \int_{Y_-(C)}^{Y_+(C)} \frac{dy}{X^*_+(y)}, \qquad T_2(C) = \int_{Y_-(C)}^{Y_+(C)} \frac{dy}{|X^*_-(y)|}$$

$$T = T(C) = T_1(C) + T_2(C)$$

Define a $T(C)$-periodic function $\Psi(t) = \Psi_C(t)$, $0 \leqslant t < \infty$, as the continuous solution of the equation

$$\dot{\Psi}_C(t) = \begin{cases} X^*_+(\Psi_C(t)), & 0 \leqslant t < T_1(C) \\ X^*_-(\Psi_C(t)), & T_1(C) \leqslant t < T(C) \end{cases}$$

$$\Psi_C(0) = Y_-(C)$$

for $t \in [0, T)$, and let

$$\Phi(t) = \Phi_C(t) = \dot{\Psi}_C(t)$$

The function $\Phi_C(t)$ has discontinuities at points $0, \pm T, \pm 2T, \ldots$.

Theorem 1. Let Conditions 1 and 2 be satisfied. Let

$$\varepsilon = \varepsilon(\delta) = \frac{C\delta}{\ln \delta^{-1}}$$

where C is a positive constant, and $Y^{\varepsilon(\delta)}_t$ be the solution of (2) with initial conditions $Y^{\varepsilon(\delta)}_0 = y$, $\dot{Y}^{\varepsilon(\delta)}_0 = x$.

1. If $C < (\bar{V}_+ \wedge \bar{V}_-)$ and $x < X^*_0(y)$, then for any $A, h > 0$,

$$\lim_{\delta \downarrow 0} P_{x,y}\{\max_{0 \leqslant t \leqslant A} |Y^{\varepsilon(\delta)}_t - \bar{Y}^y_-(t)| < h\} = 1 \tag{6}$$

$$\lim_{\delta \downarrow 0} P_{x,y}\left\{\int_0^A |\dot{Y}^{\varepsilon(\delta)}_t - X^*_-(\bar{Y}^y_-(t))|^2\, dt < h\right\} = 1 \tag{7}$$

If $x > X^*_0(y)$, then (6) and (7) hold with replacement of $\bar{Y}^y_-(t)$ by $\bar{Y}^y_+(t)$ and $X^*_-(y)$ by $X^*_+(y)$.

2. If $\bar{V}_+ < C < \bar{V}_-$, then (6) and (7) hold for any x and y. If $\bar{V}_- < C < \bar{V}_+$, then (6) and (7) hold with $\bar{Y}^y_-(t)$ and $X^*_-(y)$ replaced by $\bar{Y}^y_+$ and $X^*_+(y)$.

3. If $(\bar{V}_+ \vee \bar{V}_-) < C < \Lambda$ and $Y_0^\varepsilon = y = Y_-(C)$, then for any $A, h > 0$

$$\lim_{\delta \downarrow 0} P_{x,y}\{\max_{0 \leqslant t \leqslant A} |Y_t^{\varepsilon(\delta)} - \Psi_C(t)| > h\} = 0$$
$$\lim_{\delta \downarrow 0} P_{x,y}\left\{\int_0^A |X_t^{\varepsilon(\delta)} - \Phi_C(t)|^2 \, dt > h\right\} = 0 \tag{8}$$

uniformly in x from any compact subset of $\mathbb{R}^1$. If $y \neq Y_-(C)$ then, first, the trajectory $Y_t^{\varepsilon(\delta)}$ will come to $Y_-(C)$ along $\bar{Y}_0^y(t)$ or $\bar{Y}_+^y(t)$ and then $(X_t^{\varepsilon(\delta)}, Y_t^{\varepsilon(\delta)})$ will be close to the periodic function $(\Phi_C(t), \Psi_C(t))$ with an appropriate phase shift.

4. If $C \geqslant \Lambda = V_\pm(y^*)$ and $Y_0^\varepsilon = y = y^*$, then for any $A, h > 0$

$$\lim_{\delta \downarrow 0} P_{x,y}\{\max_{0 \leqslant t \leqslant A} |Y_t^{\varepsilon(\delta)} - y^*| > h\} = 0$$

uniformly in x from any compact subset of $\mathbb{R}^1$. If $y \neq y^*$, then the trajectory $(X_t^{\varepsilon(\delta)}, Y_t^{\varepsilon(\delta)})$, first, will come to y^* along $\bar{Y}_-^y(t)$ or $\bar{Y}_+^y(t)$ and then stay in a small neighborhood of y^*: for any $A, h > 0$ and any $x, y \in \mathbb{R}^1$ there exists $\hat{t} = \hat{t}(y)$ such that

$$\lim_{\delta \downarrow 0} P_{x,y}\{\max_{\hat{t} \leqslant t \leqslant A} |Y_t^{\varepsilon(\delta)} - y^*| > h\} = 0$$

I will give a sketch of the proof of this theorem. The full proof can be easily reconstructed from this sketch and the large derivation estimates from (ref. 3, Chaps. 4 and 6).

Make a time change:

$$\tilde{X}_t^\varepsilon = X_{\delta t}^\varepsilon, \quad \tilde{Y}_t^\varepsilon = Y_{\delta t}^\varepsilon, \quad \tilde{X}_t^{\varepsilon, y} = X_{\delta t}^{\varepsilon, y}$$

Then we derive from (3) and (4) that

$$\dot{\tilde{X}}_t^\varepsilon = f(\tilde{X}_t^\varepsilon, \tilde{Y}_t^\varepsilon) + \sqrt{\varepsilon'}\, \sigma(\tilde{X}_t^\varepsilon, \tilde{Y}_t^\varepsilon) \circ \dot{W}_t$$
$$\dot{\tilde{Y}}_t^\varepsilon = \delta \tilde{X}_t^\varepsilon, \qquad \tilde{X}_0^\varepsilon = x, \qquad \tilde{Y}_0^\varepsilon = y, \qquad \varepsilon' = \frac{\varepsilon(\delta)}{\delta} \tag{9}$$

and for $\tilde{X}_t^{\varepsilon, y}$ we have an equation

$$\dot{\tilde{X}}_t^{\varepsilon, y} = f(\tilde{X}_t^{\varepsilon, y}, y) + \sqrt{\varepsilon'}\, \sigma(\tilde{X}_t^{\varepsilon, y}, y) \circ \dot{W}_t$$
$$\tilde{X}_0^{\varepsilon, y} = x \tag{10}$$

Since $\varepsilon(\delta) = C\delta/(\ln \delta^{-1})$, we have: $\varepsilon' = \varepsilon'(\delta) = C/(\ln \delta^{-1})$.

Let D_1^y (D_2^y) be the basin of $X_-^*(y)$ (of $X_+^*(y)$):

$$D_1^y = \{x < X_0^*(y)\}, \qquad D_2^y = \{x > X_0^*(y)\}$$

Define exit times

$$\tilde{\tau}_i^y = \min\{t : \tilde{X}_t^{\varepsilon, y} \notin D_i^y\}, \qquad i \in \{1, 2\}$$

It follows from Theorem 4.4.2 from ref. 3, that for any $h > 0$

$$\lim_{\delta \downarrow 0} P_x \left\{ \exp\left\{\frac{V_+(y) - h}{\varepsilon'(\delta)}\right\} < \tilde{\tau}_2^y < \exp\left\{\frac{V_+(y) + h}{\varepsilon'(\delta)}\right\}\right\}$$
$$= \lim_{\delta \downarrow 0} P_x\{\delta^{-(V_+(y) - h)/C} < \tilde{\tau}_2^y < \delta^{-(V_+(y) + h)/C}\} = 1 \qquad (11)$$

for $x \in D_2^y$,

$$\lim_{\delta \downarrow 0} P_x\{\delta^{-(V_-(y) - h)/C} < \tilde{\tau}_1^y < \delta^{-(V_-(y) + h)/C}\} = 1$$

for $x \in D_1^y$. Since $\tau_i^y = \min\{t : X_t^{\varepsilon, y} \notin D_i\}$ is distributed as $\delta \tilde{\tau}_i^y$, $i \in \{1, 2\}$, (11) implies:

$$\lim_{\delta \downarrow 0} P_x\{\delta^{(C - V_+(y) + h)/C} < \tau_2^y < \delta^{(C - V_+(y) - h)/C}\} = 1, \qquad x \in D_2^y$$
$$\lim_{\delta \downarrow 0} P_x\{\delta^{(C - V_-(y) + h)/C} < \tau_1^y < \delta^{(C - V_-(y) - h)/C}\} = 1, \qquad x \in D_1^y \qquad (12)$$

Since (12) is satisfied for any $h > 0$, we conclude that the process $X_t^{\varepsilon, y}$, $X_0^{\varepsilon, y} = x \in D_1$, defined by (4), leaves D_1 immediately if $C > V\_(y)$ and $\delta \downarrow 0$. If $C < V\_(y)$, then $P_x\{\tau_1^y > M\} \to 1$ as $\delta \downarrow 0$ for any M and $x \in D_1$. A similar statement holds for the exit time τ_2^y from D_2.

On the other hand, when the process $X_t^{\varepsilon, y}$ stays in D_1 (in D_2), it spends most of the time in a small neighborhood of $X_-^*(y)$ (of $X_+^*(y)$). This follows from Theorem 4.4.3 of ref. 3. Taking into account smoothness of $X_\pm^*(y)$ and slowness of the Y-component in (9), one can derive that, for any $h > 0$, the process $(X_t^{\varepsilon(\delta)}, Y_t^{\varepsilon(\delta)})$ spends most of the time in the h-neighborhood of the curves $X_+^*(y)$ or $X_-^*(y)$ if $\delta > 0$ is small enough. This implies that when $X_t^{\varepsilon(\delta)}$ is in D_2 (D_1) the evolution of $Y_t^{\varepsilon(\delta)}$ is close as $\delta \downarrow 0$ to the deterministic evolution governed by the first or second of equations (5), depending on the sign of the difference $X_t^{\varepsilon(\delta)} - X_0^*(Y_t^{\varepsilon(\delta)})$.

Equalities (12) together with the slowness of the Y-component imply that $X_t^{\varepsilon(\delta)}$, with probability close to 1, when δ is small, "jumps" to D_2 if $Y_t^{\varepsilon(\delta)} < Y^-(C)$ and to D_1 if $Y_t^{\varepsilon(\delta)} > Y^+(C)$.

These arguments allow to prove Theorem 1.

A similar result holds if the function $f(\dot{y}, y)$ in Eq. (1) behaves like in Fig. 1. In this case, the functions $V_{\pm}(y)$ are well defined for $y \in [m, M]$, where m is the local minimum of the curve $Y(\dot{y})$ in Fig. 1, and M is the local maximum, $V_{+}(M)=0$, $V_{-}(m)=0$. Assume additionally that these functions are monotone, and let $V_{+}(Y^*)=V_{-}(Y^*)=\Lambda$. Then $Y_y^{\varepsilon(\delta)}$ stabilizes to Y^* if

$$\lim_{\delta \downarrow 0} \frac{\varepsilon(\delta)}{\delta}=0, \qquad \lim_{\delta \downarrow 0} \frac{\varepsilon(\delta)}{\delta} \ln \delta^{-1} \geqslant \Lambda$$

If $\varepsilon(\delta)=C\delta(\ln \delta^{-1})^{-1}$ and $C \in (0, \Lambda)$, then a solution close to periodic with the period $T(C)$ defined in Theorem 1 will appear. These oscillations approach the loop shown in Fig. 1, when C tends to zero.

Example. Consider the linear oscillator with a friction:

$$\delta \ddot{y}=-y-\beta \dot{y}(\dot{y}^2-1)$$

Then $m=-M=2\beta/3\sqrt{3}$. Denote by $X_-^*(y)<X_0^*(y)<X_+^*(y)$ the roots of the polynomial $-\beta X^3+\beta X=y$. If $|y| \leqslant 2\beta/3\sqrt{3}$ then all three roots are real. When $\delta \ll 1$, the system has a periodic solution close to the relaxational oscillations. The point $(0, 0)$ is an unstable equilibrium point and no other equilibrium exits. Add to the equation a small white noise:

$$\delta \ddot{y}=-y-\beta \dot{y}(\dot{y}^2-1)+\sqrt{\varepsilon}\, \dot{W}_t$$

Put $\beta X^4/2-\beta X^2=G(X)$. Then

$$V_{\pm}(y)=G(X_0^*(y))-G(X_{\pm}^*(y)) \qquad \text{and} \qquad V_{+}(0)=V_{-}(0)=\frac{\beta}{2}=\Lambda$$

If $\beta\delta/(2\ln \delta^{-1}) \leqslant \varepsilon(\delta) \ll \delta$, then the unstable for the non-perturbed system equilibrium at $(0, 0)$ will be, in a sense, stable for the perturbed system: For any A, $h>0$ and $|y|$ small enough,

$$P_{x,y}\{\max_{0 \leqslant t \leqslant A} |Y_t^{\varepsilon(\delta)}|<h\} \to 1$$

as $\delta \downarrow 0$. Actually, one can even choose $A=A(\delta)$ such that $A(\delta) \uparrow \infty$ as $\delta \to 0$ but not too fast.

If $\varepsilon(\delta)=C\delta/(\ln \delta^{-1})$, $0<C<\beta/2$, then the system will perform oscillations with the amplitude L which satisfies the equation $G(X_0^*(L))-G(X_+^*(L))=C$.

One can show that, due to a certain symmetry in this example, the process Y_t^ε can be stabilized at zero not just by the noise $\sqrt{\varepsilon}\,\dot{W}_t$, but also by a class of symmetric noises ζ_t^ε with the action functional $\varepsilon^{-1}\tilde{S}_{0T}(\varphi)$, where $\tilde{S}_{0T}(\varphi)$ is regular enough if

$$\lim_{\delta\downarrow 0}\frac{\varepsilon(\delta)}{\delta}=0, \qquad \lim_{\delta\downarrow 0}\frac{\varepsilon(\delta)}{\delta}\ln\delta^{-1}=\infty$$

If the functions $V_{\pm}(u)$ are not monotone, the perturbed system may have many asymptotically quasi-stable equilibriums and various oscillations induced by the noise. They can be described in an analogous way.

3. NON-GAUSSIAN PERTURBATIONS. A REMARK ON COMPUTER SIMULATIONS

1. Let $\xi_1(t)$ and $\xi_2(t)$ be independent Poisson processes with parameter $\lambda\varepsilon^{-1}$, $0<\varepsilon\ll 1$, $0<\lambda$. Put $\eta_t^\varepsilon=\varepsilon(\xi_1(t)-\xi_2(t))$. Consider perturbations of Eq. (1) by $\dot{\eta}_t^\varepsilon$

$$\delta\ddot{Y}_t^\varepsilon=f(\dot{Y}_t^\varepsilon, Y_t^\varepsilon)+\dot{\eta}_t^\varepsilon, \qquad Y_0^\varepsilon=y, \qquad \dot{Y}_0^\varepsilon=x \tag{13}$$

Now, the noise consists of a sequence of small mean zero δ-function-like impulses which appear at random times with the intensity $2\lambda\varepsilon^{-1}$. It is easy to see that η_t^ε tends to 0 as $\varepsilon\downarrow 0$ and Y_t^ε converges to Y_t. The normalized process $(1/\sqrt{\varepsilon})\,\eta_t^\varepsilon$ converges weakly on any time interval to $\sqrt{2\lambda}\,W_t$, where W_t is a Wiener process. The perturbed process is, in a sense, close to $\hat{Y}_t^\varepsilon$,

$$\delta\ddot{\hat{Y}}_t^\varepsilon=f(\dot{\hat{Y}}_t^\varepsilon, \hat{Y}_t^\varepsilon)+\sqrt{2\varepsilon\lambda}\,\dot{W}_t, \qquad \hat{Y}_0=y, \qquad \dot{\hat{Y}}_0=x \tag{14}$$

and one could expect that the results of the previous section can be used to describe Y_t^ε for $0<\varepsilon\ll\delta\ll 1$. But since the behavior of Y_t^ε on time intervals of order 1 for such ε and δ is defined by large deviations, the approximation (14) does not work.

To describe the behavior of $Y_t^{\varepsilon(\delta)}$ as $\delta\downarrow 0$, $\varepsilon(\delta)\downarrow 0$, consider the process $X_t^{\varepsilon,y}$:

$$\delta\dot{X}_t^{\varepsilon,y}=f(\dot{X}_t^{\varepsilon,y}, y)+\dot{\eta}_t^\varepsilon, \qquad X_0^{\varepsilon,y}=x \tag{15}$$

According to our reasoning in the previous section, we need the action functional[3] for the family of processes $\tilde{X}_t^{\varepsilon',y}$, $\varepsilon'=\varepsilon/\delta\downarrow 0$, where

$$\dot{\tilde{X}}_t^{\varepsilon',y}=f(\tilde{X}_t^{\varepsilon',y}, y)+\dot{\eta}_t^{\varepsilon'}, \qquad \tilde{X}_0^{\varepsilon',y}=x$$

The action functional for the Poisson family $\varepsilon' \xi_1(t)$ is $\varepsilon'^{-1} \hat{S}_{0T}(\varphi)$, $\varphi \in C_{0T}$, where (see Section 5.3 of ref. 3)

$$\hat{S}_{0T}(\varphi) = \begin{cases} \int_0^t \hat{L}(\dot{\varphi}_s)\, ds, & \varphi_s \text{ is absolutely continuous} \\ +\infty, & \text{for the rest of } C_{0T} \end{cases}$$

$$\hat{L}(\beta) = \begin{cases} \beta \ln \dfrac{\beta}{\lambda} - \beta + \lambda, & \beta \geqslant 0 \\ +\infty, & \beta < 0 \end{cases}$$

Using this and independence of $\xi_1(t)$ and $\xi_2(t)$ one can calculate the action functional for $\eta_t^\varepsilon = \varepsilon(\xi_1(t) - \xi_2(t))$ and then the action functional $(1/\varepsilon)\, S_{0T}(\varphi)$ for $\tilde{X}_t^{\varepsilon', y}$ as $\varepsilon' \downarrow 0$. For φ absolutely continuous,

$$S_{0T}(\varphi) = \int_0^T L(\varphi_s, \dot{\varphi}_s, y)\, ds$$

Actually, we need not the function $L(x, \beta, y)$, but the Legendre transformation $H(x, \alpha, y)$ of this function in β, so that I calculate just $H(x, \alpha, y)$:

$$H(x, \alpha, y) = \lambda(e^\alpha + e^{-\alpha}) - 2\lambda - \alpha f(x, y)$$

Assume that Condition 1 is satisfied. To define the counterparts of $V_\pm(y)$, we should, according to the results of Chap. 5 of ref. 3, consider the equation $H(x, du/dx, y) = 0$. A solution of this equation satisfying some additional conditions is called quasi-potential (see ref. 3, Section 5.4). Let $\alpha = G(f)$ be the inverse function to the function

$$f = \frac{\lambda(e^\alpha + e^{-\alpha}) - 2\lambda}{\alpha}$$

Then the equation for $u(x)$ has the form $du/dx = G(f(x, y))$. Consider the solution of this equation with the condition $u(X_0^*(y)) = 0$, $u(x) > 0$ for $x \neq X_0^*(y)$. It follows from ref. 3 that the amounts

$$V_\pm(y) = u(X_\pm^*(y)) = \int_{X_0^*(y)}^{X_\pm^*(y)} G(f(z, y))\, dz \tag{16}$$

characterize the exit times from the basins of $X_+^*(y)$ and $X_-^*(y)$ for the process (15) so that bounds (11) hold.

This allows to repeat the arguments used in Theorem 1 to prove the following result:

Theorem 2. Let Y_t^ε be the solution of (13) with η_t^ε introduced above and $\varepsilon = \varepsilon(\delta) = C\delta/(\ln \delta^{-1})$. Let functions $V_\pm(y)$ be defined by (16). Assume that Condition 1 is satisfied and the functions $V_\pm(y)$ satisfy Condition 2. Then the statement of Theorem 1 holds.

One can consider more general impulse perturbations using results of Chap. 5 of ref. 3.

2. Let ν_t be a continuous time Markov chain. Assume for brevity that $\nu(t)$ has just two states, 1 and 2, and that the transition intensities matrix is equal to

$$\begin{pmatrix} -q & q \\ q & -q \end{pmatrix}$$

Then, of course, the uniform distribution $(\frac{1}{2}, \frac{1}{2})$ is invariant.

Consider a continuous solution of a differential equation

$$\delta \ddot{Y}_t^\varepsilon = f_{\nu(t/\varepsilon)}(\dot{Y}_t^\varepsilon, Y_t^\varepsilon), \qquad Y_0^\varepsilon = y, \qquad \dot{Y}_0^\varepsilon = x \tag{17}$$

where $f_1(x, y) > 0$ and $f_2(x, y) < 0$ and

$$\tfrac{1}{2}[f_1(x, y) + f_2(x, y)] = f(x, y)$$

is the same as in (1). It is easy to prove that Y_t^ε converges in probability as $\varepsilon \downarrow 0$, uniformly on any finite time interval, to the solution of (1) so that (17) can be considered as a perturbation of (1). Again one can prove that

$$\left(\frac{Y_t^\varepsilon - Y_t}{\sqrt{\varepsilon}}, \frac{\dot{Y}_t^\varepsilon - \dot{Y}_t}{\sqrt{\varepsilon}} \right)$$

converges weakly as $\varepsilon \downarrow 0$ to a Gaussian process. And again this normal approximation does not describe the transitions between the neighborhoods of $X_1^*(y)$ and $X_2^*(y)$.

To describe the behavior of solutions of (17), one should, first, consider the equation

$$\dot{X}_t^{\varepsilon, y} = \frac{1}{\delta} f_{\nu(t/\varepsilon)}(X_t^{\varepsilon, y}, y), \qquad X_0^{\varepsilon, y} = x$$

with the frozen y. Large deviations for the process $\tilde{X}_t^{\varepsilon', y} = X_{\delta t}^{\varepsilon, y}$, $\varepsilon' = (\varepsilon/\delta) \to 0$, are described in Chap. 7 of ref. 3. One should calculate

$$\lim_{T \to \infty} \frac{1}{T} \ln E \exp\left\{ \alpha \int_0^T f_{\nu(s)}(x, y)\, ds \right\} = H(\alpha, x, y) \tag{18}$$

If $v(t)$ is the introduced above Markov chain, then $H(\alpha, x, y)$ is equal (Theorem 7.7.2 from ref. 3) to the eigenvalue of the matrix

$$\begin{pmatrix} -q+\alpha f_1(x, y) & q \\ q & -q+\alpha f_2(x, y) \end{pmatrix}$$

with the maximal real part. Such an eigenvalue is real and equal to

$$H(\alpha, x, y)=\alpha \bar{f}(x, y)-q+\sqrt{\alpha^2 \underline{f}^2(x, y)+q^2}$$

where $\bar{f}(x, y)=\frac{1}{2}(f_1(x, y)+f_2(x, y))$, $\underline{f}(x, y)=\frac{1}{2}(f_1(x, y)-f_2(x, y))$.

Let $u(x)$ be the solution of the equation

$$H\left(\frac{du}{dx}, x, y\right)=0, \qquad u(X_0^*(y))=0$$
$$u(x)>0 \qquad \text{for} \quad x \neq X_0^*(y) \tag{19}$$

Then the function $u(x)$ is a kind of the quasi-potential for this type of perturbations. Solving problem (19), we find that

$$u(x, y)=q \int_{X_0^*(y)}^{x} \frac{f_1(z, y)+f_2(z, y)}{f_1(z, y)\, f_2(z, y)}\, dz$$

Put

$$V_{\pm}(y)=u(X_{\pm}^*(y), y) \tag{20}$$

Using the same arguments as in Theorem 1 and large deviation bounds obtained in Chap. 6 of ref. 3 we can prove the following result.

Theorem 3. Let Y_t^ε be the solution of Eq. (17) and functions $V_+(y)$ and $V_-(y)$ be defined by (20). Assume that Conditions 1 and 2 are satisfied, and let $\varepsilon=\varepsilon(\delta)=C\delta(\ln \delta^{-1})^{-1}$. Then the statement of Theorem 1 holds.

One can use the same construction if v_t is a Markov process with any compact phase space $\mathscr{E}$. Under certain mild conditions, the limit (18) exists and is equal to the first eigenvalue $H(\alpha, x, y)$ of the operator A^α: $A^\alpha g(z)=Ag(z)+\alpha f(x, y, z)\, g(z)$, $z \in \mathscr{E}$, where A is the generator of the process v_t.

This approach can be used also if v_t is not a Markov process but a stationary process with strong enough mixing properties.

3. Consider Eq. (2) with $\sigma(\dot{y}, y) \equiv 1$:

$$\delta \ddot{Y}_t^\varepsilon = f(\dot{Y}_t^\varepsilon, Y_t^\varepsilon) + \sqrt{\varepsilon}\, \dot{W}_t$$

To describe the asymptotics of Y_t^ε as ε, δ tend to zero, we should consider large deviations for the process $\tilde{X}_t^{\varepsilon', y}$ introduced in Section 2:

$$\dot{\tilde{X}}_t^{\varepsilon', y} = f(\tilde{X}_t^{\varepsilon', y}, y) + \sqrt{\varepsilon'}\, \dot{W}_t \tag{21}$$

where $\varepsilon' = \varepsilon/\delta$. Computer simulations represent an essential part of research related to stochastic-resonance-type effects: the Wiener process multiplied by $\sqrt{\varepsilon'}$ in (21) is replaced by a symmetric random walk $\xi_t^{\tau, h}$ with time unit τ and space unit h, and the equation (21) is replaced by the corresponding difference equation.

Since the diffusion coefficient in (21) is equal to ε'; the time and space steps should satisfy the equality $h^2/\tau = \varepsilon'$.

But the large deviations for the random walk $\xi_t^{\tau, h}$ and $\sqrt{\varepsilon'}\, W_t$ are, in general, different, so that one should make special arrangements to make this approximation working. In our problems the quasi-potential which defines the functions $V_\pm(y)$ is important. In the case of process (21) the quasi-potential, actually, is the classical potential of the field $f(x, y)$ (up to a constant factor), and it is defined as the antiderivative in x of the function $-2f(x, y)$, y is a parameter.

To calculate the quasi-potential for the corresponding difference equation when $\sqrt{\varepsilon}\, W_t$ is replaced by a symmetric random walk $\xi_t^{\tau, h}$ which makes one step to the right or to the left in a time unit, one should consider the function

$$\begin{aligned} G^{\tau, h}(x, y, z) &= zf(x, y) + \frac{1}{\tau} \ln E_x \exp\{z(\xi_t^{\tau, h} - x)\} \\ &= zf(x, y) + \frac{1}{\tau} \ln \left(\frac{e^{hz} + e^{-hz}}{2} \right) \end{aligned}$$

Let $\tau/h^2 = 1/\varepsilon'$. Define $\mu = \tau/h$,

$$H(x, y, z) = zf(x, y) + \mu^{-2} \ln \frac{e^{\mu z} + e^{-\mu z}}{2}$$

Then as one can derive from ref. 4, Sections 3.2 and 4.2, the equation for the quasipotential $u(x)$ has the form $H(x, y, du/dx) = 0$.

Let $Z = Z_\mu(f)$ be the function inverse to

$$f = f(z) = -\frac{1}{z\mu^2} \ln \frac{e^{\mu z} + e^{-\mu z}}{2} \tag{22}$$

Then

$$u(x) = u(x, y) = \int_{x_0}^{x} Z_\mu(f(z, y))\, dz$$

For $0 < \mu \ll 1$, we derive from (22):

$$f = f(z) = -\frac{1}{2} z + \frac{\mu^2 z^3}{12} + o(\mu^2) \tag{23}$$

One can see from (23) that the zero approximation $Z_\mu^0(f)$ for $Z_\mu(f)$ is given as $Z_\mu^0(f) = -2f$. So that if $\tau/h^2 = 1/\varepsilon'$ and $\mu = \tau/h$ is small, the results of simulations will be close to the real behavior of the process $\tilde{X}_t^{\varepsilon, y}$ defined by (21).

We can calculate also the correction term which compensates the replacement of the diffusion by random walk. Equation (23), implies that a more precise expression for the function $Z_\mu(f)$ is given by the formula

$$Z_\mu(f) = -2f - \tfrac{4}{3}\mu^2 f^3 + o(\mu^2)$$

This means that a more precise expression for the quasi-potential corresponding to the random walk with $\tau/h^2 = 1/\varepsilon'$ and $\tau/h = \mu$, $0 < \mu \ll 1$, is given by the formula

$$u(x) = -2 \int_{x_0}^{x} f(z, y)\, dz - \frac{4\mu^2}{3} \int_{x_0}^{x} f^3(z, y)\, dz$$

4. MANY DEGREES OF FREEDOM

Consider a system of n degrees of freedom:

$$\delta \ddot{Y}_t = f(\dot{Y}_t, Y_t), \qquad Y_0 = y, \qquad \dot{Y}_0 = x, \qquad x, y \in \mathbb{R}^n \tag{24}$$

Here $f(x, y)$ is an n-dimensional vector smoothly depending on $x, y \in \mathbb{R}^n$, $0 < \delta \ll 1$. To be specific we examine the perturbations of (24) by the standard n-dimensional Gaussian white noise:

$$\delta \ddot{Y}_t^\varepsilon = f(\dot{Y}_t^\varepsilon, Y_t^\varepsilon) + \sqrt{\varepsilon}\, \dot{W}_t, \qquad \dot{Y}_0^\varepsilon = x, \quad Y_0^\varepsilon = y \tag{25}$$

Together with (25), consider the process $X_t^{\varepsilon, y}$, $y \in \mathbb{R}^n$:

$$\delta \dot{X}_t^{\varepsilon, y} = f(X_t^{\varepsilon, y}, y) + \sqrt{\varepsilon}\, \dot{W}_t, \qquad X_0^{\varepsilon, y} = x$$

Assume that, for any $y \in \mathbb{R}^n$, the vector field $f(x, y)$ has two asymptotically stable equilibriums $X_1^*(y)$ and $X_2^*(y)$ and a saddle point $X_0^*(y)$ separating them. Let $\Gamma = \Gamma_y$ be the separatrice surface dividing $\mathbb{R}^n$ into two parts: D_1 which is attracted to $X_1^*(y)$, and D_2 which is attracted to $X_2^*(y)$, $D_1 \cup D_2 \cup \Gamma = \mathbb{R}^n$. Let the functions $X_1^*(y)$, $X_2^*(y)$, $X_0^*(y)$ be twice continuously differentiable. We refer to these assumptions as Condition 3.

Recall the notion of quasi-potential of the vector field $b(x)$ in $\mathbb{R}^n$ with respect to the white noise perturbations:[3] A twice continuously differentiable function $U(x)$, $x \in \mathbb{R}^n$, is called the quasi-potential of the field $b(x)$, if the field $l(x) = b(x) + \nabla U(x)$ is orthogonal to $\nabla U(x)$ for each $x \in \mathbb{R}^n$.

If the field $b(x)$ is potential, $b(x) = -\nabla U(x)$, then the potential $U(x)$ coincides with the quasi-potential and $l(x) \equiv 0$. The orthogonality of $l(x)$ and $\nabla U(x)$ gives an equation for the quasi-potential:

$$|\nabla U(x)|^2 + b(x) \cdot \nabla U(x) = 0$$

where $b(x) \cdot \nabla U(x)$ means the scalar product in $\mathbb{R}^n$. One should mention that the smooth quasi-potential exists not always: Eq. (26) may have only generalized solution. One can always introduce the quasi-potential (Lipschitz continuous but not necessarily smooth) as the solution of a variational problem for the action functional (see ref. 3, Section 4.2).

Condition 4. Assume that the field $f(x, y)$ has a quasi-potential $U(x, y)$ for each $y \in \mathbb{R}^n$, and $U(x, y)$ is twice continuously differentiable in $x, y \in \mathbb{R}^n$, $\nabla_x U(X_i^*(y), y) = 0$, $i \in \{0, 1, 2\}$, and $\nabla_x U(x, y) \neq 0$ if $x \neq X_i^*(Y)$.

Put

$$V_i(y) = 2(U(X_0^*(y)) - U(X_i^*(y))), \qquad i \in \{1, 2\}$$

Let, as before,

$$\varepsilon = \varepsilon(\delta) = \frac{C\delta}{\ln \delta^{-1}}$$

Denote by $\tau_i = \tau_i^{\varepsilon, y}$ the first exit time from D_i, $i \in \{1, 2\}$: $\tau_i = \min\{t\colon X_t^{\varepsilon, y} \notin D_i\}$.

It follows from the results of Section 4.2 of ref. 3 and considerations of Section 2, that for any $h > 0$ and $X_0^{\varepsilon, y} = x \in D_i$, that

$$\lim_{\delta \downarrow 0} P_x\{\delta^{(C-2V_i(y)+h)/C} < \tau_i^{\varepsilon, y} < \delta^{(C-2V_i(y)-h)/C}\} = 1 \tag{26}$$

Consider two dynamical systems in $\mathbb{R}^n$:

$$\dot{Y}_i(t) = X_i^*(Y_i(t)), \qquad Y_i(0) = y, \quad i \in \{1, 2\} \tag{27}$$

Since the dynamical systems in $\mathbb{R}^n$, $n > 1$, may have much more diverse behavior than in $\mathbb{R}^1$, we restrict ourselves just to some sufficient conditions for the existence of a noise induced close-to-periodic solution.

Condition 5. Suppose that a closed domain $\Pi \subset \mathbb{R}^n$ exists such that it is bounded by $\gamma_1 \subset \{y \in \mathbb{R}^n : V_2(y) = C\}$, by $\gamma_2 \subset \{y \in \mathbb{R}^n : V_1(y) = C\}$, and by the side surface S (Fig. 4). Assume that γ_1, γ_2 and S are smooth manifolds with an edge. Let γ_1 and γ_2 be homeomorphic to the $(n-1)$-dimensional ball. Let $M = \Pi \cap \{y : V_1(y) = V_2(y)\}$ be also a smooth manifold with an edge, and let M be situated between γ_1 and γ_2 as in Fig. 4, $M \cap \gamma_i = \varnothing$, $i \in \{1, 2\}$. Let $V_2(y) > V_1(y)$ above M and $V_2(y) < V_1(y)$ below M, $V_1(y) < C$ above γ_1 and $V_1(y) > C$ below γ_1, and $V_2(y) < C$ below γ_2 and $V_2(y) > C$ above γ_2.

Assume that the trajectories $Y_1(t)$, defined by (27), enter the domain Π through γ_2 and S and leave Π through γ_1. Assume that $Y_2(t)$ enter Π through γ_1 and S and leave Π through γ_2.

Suppose that Conditions 3, 4, and 5 are satisfied. For a point $z \in \gamma_2$, consider the trajectory $Y_1(t) = Y_1(z, t)$, $Y_1(z, 0) = z$. Let

$$\tau_1 = \tau_1(z) = \min\{t : Y_1(z, t) \in \gamma_1\}$$

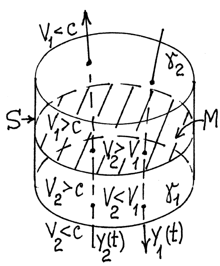

Fig. 4.

Due to Condition 5, $\tau_1(z) < \infty$ for any $z \in \gamma_2$. Denote by $Y_2(Y_1(z, \tau), t)$ the trajectory of the system $Y_2(t)$ with the initial condition $Y_2(Y_1(z, \tau_1), 0) = Y_1(z, \tau_1) \in \gamma_1$. Let

$$\tau_2 = \tau_2(z) = \min\{t : Y_2(Y_1(z, \tau_1), t) \in \gamma_2\}$$

Consider a transformation T of γ_2: $Tz = Y_2(Y_1(z, \tau_1), \tau_2)$. It follows from our assumptions that T is a transformation of γ_2 in itself. Assume that T is continuous. Since γ_2 is homeomorphic to a ball, there is a point $z^* \in \gamma_2$ such that $Tz^* = z^*$. Define $[\tau_1(z^*) + \tau_2(z^*)]$-periodic function $\Psi(t)$:

$$\Psi(t) = \begin{cases} Y_1(z^*, t), & 0 \leqslant t < \tau_1(z^*) \\ Y_2(Y_1(z^*, \tau_1), t - \tau_1), & \tau_1(z^*) \leqslant t < \tau_1(z^*) + \tau_2(z^*) \end{cases}$$

Put

$$\Phi(t) = \begin{cases} X_1^*(\Psi(t)), & 0 \leqslant t < \tau_1(z^*) \\ X_2^*(\Psi(t)), & \tau_1(z^*) \leqslant t < \tau_1(z^*) + \tau_2(z^*) \end{cases}$$

and define $\Phi(t)$ for all $t \geqslant 0$ as $(\tau_1(z^*) + \tau_2(z^*))$-periodic function.

Consider the trajectory $(\dot{Y}_t^{\varepsilon(\delta)}, Y_t^{\varepsilon(\delta)})$ starting from the point $(X_1^*(z^*), z^*)$. Then the component $\dot{Y}_t^{\varepsilon}$ will be close to $X_1^*(\Psi(t))$ in the $L^2_{[0, \alpha]}$-norm and Y_t^{ε} will be close to Ψ_t uniformly on $[0, \alpha]$ for any $\alpha \in [0, \tau_1(z^*)]$, if δ is small enough. This can be derived from (26), if one takes into account that outside of any neighborhood of the curves $X_i^*(y)$, $i \in \{0, 1, 2\}$, the component $\dot{Y}_t^{\varepsilon(\delta)}$ changes much faster then the $Y_t^{\varepsilon(\delta)}$-component. After $\Psi(t)$ crosses γ_1, $X_t^{\varepsilon} = \dot{Y}_t^{\varepsilon(\delta)}$ will "jump" to a neighborhood of $X_2^*(\Psi(t))$ with probability close to 1 as δ is small, and the evolution of Y_t^{ε} for $t > \tau_1(z^*)$ will be close to $Y_2(Y_1(z^*, \tau_1), t - \tau_1)$ until the time $\tau_1(z^*) + \tau_2(z^*)$, when X_t^{ε} will again switch to $X_1^*(\Psi(t))$, and so on. This implies the following result:

Theorem 4. Let Conditions 3, 4, and 5 be satisfied and $\varepsilon = \varepsilon(\delta) = C\delta(\ln \delta^{-1})^{-1}$. Then there exists a fixed point $z^* \in \gamma_2$ of the transformation T, and for any A, $h > 0$

$$\lim_{\delta \downarrow 0} P_{X_1^*(z^*), z^*} \left\{ \max_{0 \leqslant t \leqslant A} |Y_t^{\varepsilon(\delta)} - \Psi_t| + \int_0^A |\dot{Y}_t^{\varepsilon(\delta)} - \Phi_t|^2 \, dt > h \right\} = 0 \tag{28}$$

Theorem 4 provides conditions for process $(\dot{Y}_t^{\varepsilon}, Y_t^{\varepsilon})$ to be close to the periodic function (Φ_t, Ψ_t) on the time interval $[0, A]$ with probability close to 1 as $0 < \delta \ll 1$. One can give some additional conditions which guarantee stability of the oscillations (Φ_t, Ψ_t).

One can prove (28) with the norm of the difference $\dot{Y}^\varepsilon_t - \Phi_t$ different from $L^2_{[0,A]}$-norm. But, in general, one can not take the uniform norm.

Let now

$$\varliminf_{\delta \downarrow 0} \frac{\varepsilon(\delta) \ln \delta^{-1}}{\delta} = C > \sup_{y \in M} V_1(y)$$

This is a counterpart of part 4 of Theorem 1. Assume that the trajectories $Y_2(t)$ cross M from below upward (see Fig. 4), and $Y_1(t)$ cross M in the opposite direction. Moreover, let the projections $\pi_1(y)$ and $\pi_2(y)$ of the fields $X_1^*(y)$ and $X_2^*(y)$ on M, $y \in M$, be directed inside M on the boundary ∂M of M. Then Y^ε_t, $Y^\varepsilon_0 = y \in M$, stays in a small neighborhood of M during any time interval $[0, A]$ with probability close to 1 as $0 < \delta \ll 1$. The motion Y^ε_t will be close to a dynamical system in M corresponding to a vector field $B(y)$, $y \in M$. The field $B(y)$ is a linear combination of the fields $\pi_1(y)$ and $\pi_2(y)$. It is a delicate question to calculate the weights with which $\pi_1(y)$ and $\pi_2(y)$ are included in this combination, and we are going to address this question elsewhere. If $B(y)$ has a stable equilibrium $q \in M$, then one can expect that the component Y^ε_t, $Y^\varepsilon_0 \in M$ stabilizes to q for $0 < \delta \ll 1$ in the sense of part 4 of Theorem 1.

Finally, I would like to note that just the case when $f(x, y)$, for each $y \in \mathbb{R}^n$, has not more than 2 stable equilibriums is considered here. Existence of many stable attractors leads to a number of interesting effects. We consider this case in ref. 5. Random perturbations of more general dynamical systems with fast and slow components are considered in ref. 5 as well.

ACKNOWLEDGMENTS

This work was supported in part by NSF and by US Department of Energy.

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